Problem solving using GeoGebra -
- the case of geometry

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ABSTRACT

Problem solving is an important component of teaching mathematics. It is a way to present mathematics and it is also a skill which enhances logical reasoning. In this paper, two different approaches towards the problem solving will be presented. The first one uses classical method and the other one uses GeoGebra. The steps of both solutions of several math problems of different levels will be comparatively analyzed in order to shed light on the specific benefits of each. The group of students from the third grade, who are familiar with GeoGebra, took part in a number of exercises and their experience in this specific problem solving will also be presented.

Geometry problems of different levels have been considered in this work. The seventeen-year-olds chosen for this study are those who are a bit more interested in mathematics than the others. They are studying in the third grade (out of four) of mechanical engineering high (secondary) school in Novi Sad.

Keywords

Math teaching, logical reasoning, problem solving, GeoGebra

INTRODUCTION

Problem solving is an important component of teaching mathematics. It is a way to present mathematics and it is also a skill which enhances logical reasoning. It is established that the highest quality of knowledge is reached when students acquire it through solving a problem. The strongest connection between old and new knowledge is a well chosen problem which students solve successfully or at list try to solve. Furthermore, developing students’ skill for problem solving is a permanent task for every teacher and it is important to use benefits provided by DGS.

In this paper the two different approaches towards the problem solving will be presented. The first one uses a classical method and the other one uses GeoGebra. The steps of both solutions of several math problems of different levels will be comparatively analyzed in order to shed light on specific benefits of each. Some of
usual heuristic strategies get a new dimension in a GeoGebra environment and some of problem solving methods emerge as frequently used. In this paper, I tried to illustrate the facts considering problems of different levels and several mathematics topics (such as geometry, analytic geometry or algebra) but, in the paper, only the case of geometry will be presented.

THE PARTICIPANTS

The seventeen-year-olds chosen for this study are those who are a bit more interested in mathematics than the others. They are studying in the third grade (out of four) of mechanical engineering high (secondary) school in Novi Sad. At the time when they were doing these exercises, they passed through all the main topics of middle school mathematics except for calculus, probability and statistics.

GEOMETRY

Problem 1. Construct a triangle given side \( c \), angle \( \alpha \) and median \( m_b \).

It is not by accident that we start with construction. According to our school curricula, in the first grade Euclidean geometry is studied with the axiomatic approach where the constructive part has proved the most difficult for students. The necessary steps of a complete solution of any constructive task are: analysis, construction, proof and discussion. Those four steps correspond to Polya’s problem solving steps in such a way that analysis overlaps with understanding a problem and devising a plan, construction with carrying out the plan and finally proof and discussion with looking back.

In GeoGebra each of Euclid’s axioms is embedded into the basic commands available to the user. However, GeoGebra does not act like a pure axiomatic system because a number of basic theorems is also built in it so the environment is suitable for managing geometric objects at any chosen level. Performing basic constructions through basic commands, one can simulate the classical way of constructional task. Inherently, we can accept construction protocol as the student’s solution of a problem. Let us compare the steps of classical and GeoGebra solution of given problem.

**Solution (classical):**

*Analysis:*

- With given elements we can construct triangle \( \text{ABM} \) (SSA congruence criterion) under some conditions.

Fig. 1 Sketch with marked given elements
Construction: (description)
1. Step : construct angle $\angle Apq$ congruent with $\alpha$ (same as given)
2. Step: circle with centre in A and radius $c$ intercept row $q$ at B
3. Step: circle with centre in C and radius $t_b$ intercept (or not) leg $p$ at D (D, D1)
4. Step: circle with centre in D and radius AD (AD, AD1) intercept $p$ at C (C,C₁)
5. Step: connect points D and C

Proof: Follows from SSA congruence criterion

Discussion: (Often is the most difficult part of the solution for the students because they intuitively fix the given elements so as to get “perfect” solutions and usually overlook the special scenarios.)

The problem may have one or two solutions if the circle from step 3 intercepts row $p$ at one or two points respectively. The problem does not have a solution if the circle from step 3 does not intercept row $p$.

We can also conclude that if angle $\alpha$ is obtuse or right then the solution(s) is possible only if $m_b>c$. Also, if circle from step 3 intercept row $p$ at one point, $p$ becomes the tangent of a circle. Consequently, $m_b$ simultaneously becomes the altitude of the triangle, which means that the triangle becomes isosceles.

Solution (GeoGebra):

Analysis: In GeoGebra one can use a dynamic sketch to consider different types of triangles. The sketch can also be drawn as a solution. Another possibility is to put the given elements onto the sliders which provide better insight into the special cases.

Construction: The main part of GeoGebra solution is Construction protocol which automatically notes down the steps of the performing construction.
Fig. 2 Construction protocol as a part of a solution which can be accepted
Comparing the steps of this GeoGebra construction protocol with steps of classical
solution we can see that steps 5, 6, 7, 8, 9, 10 of this GeoGebra solution correspond to
steps 1 and 2 of the classical solution. Also, steps 11, 12, 13, 14, 15, 16 of GeoGebra
solution correspond to steps 3 and 4 of the classical solution and steps 17 and 18 of
GeoGebra solution correspond to step 5 of classical solution.

Discussion: GeoGebra makes this part more obvious by allowing the students to
manipulate objects with the use of move option; it shows the possibilities of different
scenarios of the solution(s).

Problem 2. Let $E$ be inside square $ABCD$ such that $\angle EAB = \angle EBA=15^\circ$. Show
that $\Delta CDE$ is equilateral.

This is one of my favourite examples because it has a lot of various solutions.

Fig. 3 Sketch of the problem 2.

Solution 1 (classical):

A good example for the heuristic: draw an auxiliary object
- Construct $\Delta BCG$ congruent with $\Delta ABE$.
- It is easy to notice that $\Delta BEG$ is equilateral.
- That implies $\Delta BCG \cong \Delta CGE$ (SAS congruence criterion)
- $CE \cong CB$ implies $\Delta CDE$ is equilateral

Fig. 4 Auxiliary triangle

The students like to play, and they have started to draw more auxiliary objects to
see if they can reach to the solution of their own. It was a real challenge for some of
them. There were a number of wrong attempts but one was successful.
**Solution2 (classical):** This is one student’s solution:

*Draw an auxiliary object,* again.

- Construct the square AEGF (G inside the square)
- $\triangle ADF \cong \triangle ABE$ (SAS c. criterion)
- $FG \cong FD$, $\angle DFG = 60^\circ$, $\angle DGE = 150^\circ$
- $\triangle DEG \cong \triangle ABE$ (SAS)
- $DE \cong AB$ implies $\triangle CDE$ is equilateral

![Fig.5 Auxiliary square](image)

Once, they were introduced into the problem it was a good moment to show them the richness and the power of mathematics and the importance of permanent connecting its different parts and exchanging their methods.

**Solution3 (classical):** *(Mathematical Excalibur, vol.13, number3, p.2, example 2.)*

Method: *using geometric transformation*

We introduced one very useful fact:

**Fact:** If $\alpha + \beta$ is not multiple of $360^\circ$, then $R(O_2, \beta) \circ R(O_1, \alpha) = R(O, \alpha + \beta)$,

where $\angle OO_1O_2 = \alpha/2, \angle O_2O_2O = \beta/2$

The students were asked to consider the proof of the fact as their homework.

![Fig.6 Using rotation](image)

- Let O be inside the square such that $\triangle ABO$ is equilateral
- $R(B, 30^\circ)$ sends C to O, $R(A, 30^\circ)$ sends O to D
- Since $\angle EAB = 15^\circ = \angle EBA$ according to the above fact
  $R(A, 30^\circ) \circ R(B, 30^\circ) = R(E, 60^\circ)$
- So $R(E, 60^\circ)$ sends C to D
- Therefore $\triangle CDE$ is equilateral

**Solution4 (GeoGebra)**
Teacher’s question:

- How can we use coordinates in GeoGebra to find a solution of this problem?

Fig.7 Using coordinate method

Almost all students set the origin at E. Still, a few of them were satisfied with seeing in the algebra window that all three sides of ΔCDE are of the same length.

Teacher’s questions:

- Can we accept those lengths as proof?
- Is it possible to calculate the coordinates of points and the distances between them?
- Is it OK to set the length of the side of the square?
- Which chosen length will make your calculations simpler?

After a short discussion, students agreed that the lengths could be rounded numbers and we do not know if they are really equal as well as that this is not a proof. Those questions helped them to reach the solution using the coordinate method:

- They choose the length of the side of the square to be 2
- It was easy to determine the coordinates of points:
  
  \[ A = (-1, -\tan 15^\circ), \ B = (1, -\tan 15^\circ), \ C = (1, 2 - \tan 15^\circ), \ D = (-1, 2 - \tan 15^\circ) \]

- Distance between points C and O gave the solution of this problem:
  \[
  |CO| = \sqrt{(1-0)^2 + (2-\tan 15^\circ)^2} = \sqrt{(1-0)^2 + (2-(2-\sqrt{3}))^2} = 2
  \]
- Also, by analogy they conclude \(|OD|=2\) which completes the proof.
This year the students took part in the Kangaroo contest and some of them gave the correct answer to the below stated problem.

**Problem 3.** (Kangaroo contest 2010 in Serbia)

P and Q are points on legs AC and BC of a right angled triangle, respectively. Lengths of legs are b and a. Let K and H be feet of perpendicular lines from P and Q onto hypotenuse. Which is the minimal value of KP+PQ+QH?

a) \(a + b\)  
b) \(\frac{2ab}{a+b}\)  
c) \(\frac{2ab}{\sqrt{a^2+b^2}}\)  
d) \(\frac{(a+b)^2}{\sqrt{a^2+b^2}}\)  
e) \(\frac{(a+b)^2}{2ab}\)

This is not an easy problem. When a question is followed by five suggested answers (only one correct), students usually consider the options and try to select the best of the choices offered. They use their intuition and method of elimination of the wrong answers.

Fig.8 Kangaroo Problem

This is precisely what they did with the Kangaroo Problem. None of them looked deeper into the problem and no one came to the solution by proof. That was the main reason for this problem to be chosen for these exercises with GeoGebra. The second reason is that GeoGebra is a suitable environment for considering the problems of minimum and maximum.

*Solution (GeoGebra):*

Teacher’s questions:

- If we want to consider all kinds of right angled triangles, what is the best way of making the dynamic GeoGebra sketch of it?
- Which should be our free objects?
- Which should be our dependent, but movable objects?
- Which other segments of this triangle are we interested in?
- How can we calculate the demanded value?
- With which other length can we compare the required value?
Once they made an adequate dynamic sketch of the given problem, it was easy to reach the solution. Moving points P and Q along the sides AC and BC, using algebra view, they can perceive that the length of KP+PQ+QH is minimal when points S (interception of altitude CD and segment PQ) and M (midpoint of the segment PQ) coincide.

Fig. 9 Dynamic sketch of the problem

Fig. 10 The minimal value of $KP+PQ+QH$
If points S and M coincide then CS becomes median of right angled \( \Delta PQC \) and SD becomes mid-segment of the trapezium KPQH:

\[
|CS| = \frac{|PQ|}{2} \quad \text{and} \quad |SD| = \frac{|KP| + |QH|}{2} \quad \text{or}
\]

\[
|PQ| = 2|CS| \quad \text{and} \quad |KP| + |QH| = 2|SD|
\]

Thus, the minimal length of KP+PQ+QH is 2|CS|+2|SD|=2(|CS|+|SD|)=2|CD|

Knowing CD is the altitude of right angled triangle we have: |CD| = \frac{ab}{c}

Finally minimal length of KP+PQ+QH is 2ab/c or \( 2ab / \sqrt{a^2 + b^2} \)

Finally, let us take a look at a problem which appear at IMO last year. The students were asked to use GeoGebra and find the solution on they own, as their homework.

**Problem 4.** (*IMO 2009*) Let \( ABC \) be a triangle with \( AB = AC \). The angle bisectors of \( \angle CAB \) and \( \angle ABC \) meet the sides BC and CA at D and E, respectively. Let K be the incenter of \( \Delta ADC \). Suppose that \( \angle BEK = 45^\circ \). Find all the possible values of \( \angle CAB \).

![Fig.11 IMO 2009 problem in GeoGebra](image)

**Remarks on IMO 2009** Leung Tat-Wing 2009 IMO Hong Kong Team Leader:

“This problem was also relatively easy. It is interesting to observe that an isosceles triangle can be the starting point of an IMO problem. With geometric software such as Sketchpad (or GeoGebra), one can easily see that \( \angle CAB \) should be 60° or 90°. To prove the statement of the problem, one may either use synthetic method or coordinate method. One advantage of using the coordinate method is after showing the possible values of \( \angle CAB \), one can go back to show these values do work by suitable substitutions. Some contestants lost mark either because they missed some values of \( \angle CAB \) or forgot to check the two possible cases do work.”

(*Mathematical Excalibur, Volume 14, Number 2, p.2, problem 4.*)
Joseph Louis Lagrange: “As long as algebra and geometry have been separated, their progress have been slow and their uses limited; but when these two sciences have been united, they have lent each mutual forces, and have marched together towards perfection.”

STUDENT’S COMMENTS

- With GeoGebra you can see more initial facts about the task
- With GeoGebra you can easily see if there are more solutions of the problem
- It is the best when you mix GeoGebra with classical way
- Still, GeoGebra can not do everything instead of you; you have to know theory

CONCLUSIONS

- We can accept students’ GeoGebra solutions of the constructive problem, if the construction protocol is correct
- GeoGebra fosters exchanging methods between geometry and algebra
- In GeoGebra environment coordinate method impose itself to the user as frequently applied.
- GeoGebra is the place where the borders between algebra, geometry and other mathematics topics simply disappear
- GeoGebra is a new suitable working environment for problem solving

REFERENCES


4. Mathematical Excalibur - mathematic magazine by Hong Kong University of Science and Technology, Department of Mathematics.